Unit 5 – Integration – Test 1 – Study Guide

Concepts to know:

1. Finding anti-derivatives using the reverse-chain rule (with or without u-substitution) – must know trigonometric derivatives/integrals, as well as 'e' and ln.
2. Calculating definite integrals (including integrals calculated using geometric formulas)
3. Identifying an integral as a limit of a Riemann sum.
4. Motion problems with integration
5. Estimating Integrals using Riemann sums (LRAM, RRAM, MRAM, and Trapezoidal approximations.)

Basic integrals that should be memorized:

Trigonometric Functions:

$\int \sin(u)\,du =$

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$\int \cos(u)\,du =$

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$\int \sec^2(u)\,du =$

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$\int \csc^2(u)\,du =$

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Reverse Power Rule:

$$\int x^n dx =$$

Critical Integrals to Know:

$$\int e^u du =$$

$$\int \frac{1}{u} du =$$

Evaluating a Definite Integral:

$$\int_a^b f(x)dx = F(b) - F(a) \quad \text{or} \quad \int_a^b f'(x)dx = f(b) - f(a)$$

Where F is the anti-derivative of f

Definite and Indefinite Integral Practice
If U-sub doesn't work, try algebraic manipulation and/or simplification

$$\int_0^{\sqrt{\ln x}} xe^{x^2}\,dx = \frac{x-1}{2}$$

$$\int \frac{e^x}{1+e^x}\,dx \Rightarrow \ln|1+e^x|+C$$

$$\int\left(x-\frac{1}{2x}\right)^2 dx \Rightarrow \frac{x^3}{3}-x-\frac{1}{2x^3}+C$$

$$\int_0^9 \cancel{e}^{\cancel{\ln}\sqrt{x}}\,dx = \int_0^9 \sqrt{x}\,dx = \frac{2x^{\frac{3}{2}}}{3}\Big|_0^9 = 18$$

$$\int \frac{x+1}{x^2-1}\,dx = \int \frac{\cancel{x+1}}{(\cancel{x+1})(x-1)}\,dx \Rightarrow \ln|x-1|+C$$

$$\int_0^1 \frac{x}{\sqrt{8x^2+1}}\,dx = \frac{\sqrt{8x^2+1}}{8}\Big|_0^1 = \frac{1}{4}$$

$$\int x^3\sqrt{x^4+5}\,dx \Rightarrow \frac{1}{6}\left(x^4+5\right)^{\frac{3}{2}}$$

$$\int \frac{2x^2+2x+3}{x+1}\,dx$$

$$x+1\,\overline{)\,2x^2+2x+3} \quad 2x + 3/x+1$$

$$= \int 2x+\frac{3}{x+1}\,dx \Rightarrow x^2+3\ln|x+1|+C$$

$$\int \frac{e^x}{1+3e^x}\,dx \Rightarrow \frac{1}{3}\ln|1+3e^x|+C$$

$$\int \frac{1}{x^2}\,dx = \int x^{-2}\,dx \Rightarrow -\frac{1}{x}+C$$

$$\int_1^e \frac{x^2-1}{x}\,dx = \int_1^e \frac{x^2}{x} - \frac{1}{x}\,dx$$

$$= \int_1^e x-\frac{1}{x}\,dx = \frac{x^2}{2}-\ln x\Big|_1^e = \frac{e^2-3}{2}$$

$$\int e^{2x}\sin(e^{2x}-e)\,dx \Rightarrow -\frac{1}{2}\cos(e^{2x}-e)+C$$

Multiple Choice Released AP Questions – Definite and Indefinite Integrals

$\int \cos 3x \, dx =$

(A) $-3\sin 3x + C$ (B) $-\sin 3x + C$ (C) $-\frac{1}{3}\sin 3x + C$

(D) $\frac{1}{3}\sin 3x + C$ (E) $3\sin 3x + C$

$\int \frac{1-3y}{\sqrt{2y-3y^2}} \, dy =$

(A) $4\sqrt{2y-3y^2} + C$ (B) $2\sqrt{2y-3y^2} + C$

(C) $\frac{1}{2}\ln\left(\sqrt{2y-3y^2}\right) + C$ (D) $\frac{1}{4}\ln\left(\sqrt{2y-3y^2}\right) + C$

(E) $\sqrt{2y-3y^2} + C$

$\int_1^e \frac{x^3-2}{x} \, dx =$

(A) $\frac{e^3}{3} - \frac{7}{3}$ (B) $\frac{e^3}{3} + \frac{7}{3}$ (C) $\frac{e^3}{3} - \frac{5}{3}$

(D) $\frac{e^3}{3} + \frac{5}{3}$ (E) $e^2 - \frac{2}{e}$

$\int e^x \cos\left(e^x + 1\right) dx =$

(A) $\sin\left(e^x + 1\right) + C$

(B) $e^x \sin\left(e^x + 1\right) + C$

(C) $e^x \sin\left(e^x + x\right) + C$

(D) $\frac{1}{2}\cos^2\left(e^x + 1\right) + C$

$\int_0^{\frac{\pi}{12}} \frac{dx}{\cos^2 3x} =$

(A) -3 (B) -1 (C) $-\frac{1}{3}$ (D) $\frac{1}{3}$ (E) 3

$\int_e^{e^3} \frac{\ln x}{x} \, dx =$

(A) 2 (B) $\frac{5}{2}$ (C) 4 (D) $\frac{9}{2}$ (E) 8

$$f(x) = \begin{cases} x & \text{for } x < 2 \\ 3 & \text{for } x \geq 2 \end{cases}$$

If f is the function defined above, then $\int_{-1}^{4} f(x) dx$ is

(A) $\frac{9}{2}$

(B) $\frac{15}{2}$

(C) $\frac{17}{2}$

(D) undefined

Using the substitution, $u = 3x - 1$, $\int_0^3 \sqrt{3x-1} \, dx$ is equivalent to which of the following?

(A) $\frac{1}{3}\int_{-\frac{1}{3}}^{\frac{4}{3}} \sqrt{u} \, du$ (B) $\int_{-1}^{8} \sqrt{u} \, du$ (C) $\frac{1}{3}\int_{-1}^{8} \sqrt{u} \, du$

(D) $\int_0^3 \sqrt{u} \, du$ (E) $\frac{1}{3}\int_0^3 \sqrt{u} \, du$

Using the substitution $u = \sqrt{x}$, the integral $\int_1^9 \frac{\sin\sqrt{x}}{\sqrt{x}} dx$ is equal to which of the following?

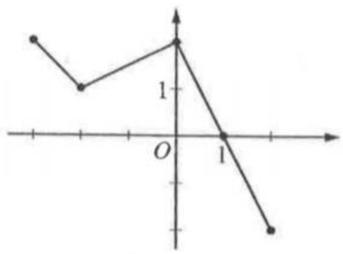
(A) $\frac{1}{2}\int_1^3 \sin u \, du$

(B) $2\int_1^3 \frac{\sin u}{u} du$

(C) $2\int_1^3 \sin u \, du$

(D) $2\int_1^9 \sin u \, du$

The Definite Integral as Area, and Integrals Using Geometric Formulas



Graph of f

The graph of a piecewise linear function $f(x)$, for $-3 \le x \le 2$, is shown above. What is the value of $\int_{-2}^{2}(f(x)+2)\,dx$?

(A) 5
(B) 6.5
(C) 11
(D) 12.5

$$f(x)=\begin{cases} \dfrac{|x-1|}{x-1} & , x \neq 1 \\ 1 & , x=1 \end{cases}$$

If f is the function defined above, then $\int_{-1}^{4} f(x)\,dx$ is

(A) 1
(B) 2
(C) 5
(D) nonexistent

The function g is continuous on the closed interval [2, 10]. If $\int_{2}^{10} g(x)\,dx = 63$ and $\int_{10}^{5} \frac{1}{2} g(x)\,dx = -16$, then $\int_{2}^{5} 2g(x)\,dx =$

(A) 31
(B) 62
(C) 95
(D) 190

$$\int_{-4}^{0} \sqrt{16-x^2}\,dx \quad = 4\pi$$

$$\int_{2\sqrt{2}}^{-2\sqrt{2}} \sqrt{8-x^2}+3\,dx \quad = -4\pi - 12\sqrt{2}$$

The Definite Integral a Limit of a Riemann Sum

Limit Statement	Definite Integral
$\lim_{n\to\infty}\sum_{k=1}^{n}\left((2+\frac{3}{n}k)^2+2\right)\cdot\frac{3}{n}$	$\int_2^5 x^2+2\,dx$
$\cancel{\lim_{n\to\infty}\sum_{k=1}^{n}\left(3(\frac{2}{n}(k-1)+6\right)\cdot\frac{2}{n}}$	
$\lim_{n\to\infty}\sum_{k=1}^{n}\left(2(4+\frac{5}{n}k)^3+6\right)\cdot\frac{5}{n}$	$\int_4^9 2x^3+6\,dx$
$\lim_{n\to\infty}\sum_{k=1}^{n}\left(4-(\frac{1}{n}k)^2\right)\cdot\frac{1}{n}$	$\int_0^1 4-x^2\,dx$
$\cancel{\lim_{n\to\infty}\sum_{k=1}^{n}\left(\sin(3+\frac{7}{n}(k-1)\right)\cdot\frac{7}{n}}$	

Definite Integral	Limit Statement
$\int_{-1}^{2}(x^2-6)dx$	$\lim_{n\to\infty}\sum_{k=1}^{n}\left[\left(-1+\frac{3k}{n}\right)^2-6\right]\frac{3}{n}$
$\int_{1}^{9}(5x+4)dx$	$\lim_{n\to\infty}\sum_{k=1}^{n}\left[5\left(1+\frac{8k}{n}\right)+4\right]\frac{8}{n}$

Which of the following limits is equal to $\int_3^7 x^3dx$?

(A) $\lim_{n\to\infty}\sum_{k=1}^{n}\left(\left(3+\frac{k}{n}\right)^3\cdot\frac{1}{n}\right)$

(B) $\lim_{n\to\infty}\sum_{k=1}^{n}\left(\left(3+\frac{k}{n}\right)^3\cdot\frac{4}{n}\right)$

(C) $\lim_{n\to\infty}\sum_{k=1}^{n}\left(\left(3+\frac{4k}{n}\right)^3\cdot\frac{1}{n}\right)$

(D) $\lim_{n\to\infty}\sum_{k=1}^{n}\left(\left(3+\frac{4k}{n}\right)^3\cdot\frac{4}{n}\right)$

Which of the following integral expressions is equal to $\lim_{n\to\infty}\sum_{k=1}^{n}\left(\left(1+\frac{2k}{n}\right)^2\cdot\frac{1}{n}\right)$?

(A) $\int_0^1(1+2x)^2\,dx$

(B) $\int_0^2(1+x)^2\,dx$

(C) $\int_1^3 x^2\,dx$

(D) $\frac{1}{2}\int_0^2 x^2\,dx$

Motion Problems Utilizing Integration

A particle moves along the x-axis with velocity given by $v(t) = 3t^2 + 6t$ for time $t \geq 0$. If the particle is at position $x = 2$ at time $t = 0$, what is the position of the particle at time $t = 1$?

(A) 4
(B) 6
(C) 0
(D) 11
(E) 12

A particle travels in a straight line with a constant acceleration of 3 meters per second per second. If the velocity of the particle is 10 meters per second at time 2 seconds, how far does the particle travel during the time interval when its velocity increases from 4 meters per second to 10 meters per second?

Very challenging!!

(A) 20 m
(B) 14 m
(C) 7 m
(D) 6 m
(E) 3 m

A particle moves along the x-axis with acceleration at any time t given as $a(t) = 3t^2 + 4t + 6$. If the particle's initial velocity is 10 and its initial position is 2, what is the position function? $s(t) = \frac{t^4}{4} + \frac{2t^3}{2} + 3t^2 + 10t + 2$

A bottle rocket is shot upward from a 10 foot stand with velocity $v(t) = 50 - 1.6t$. What is the position of the bottle rocket after 2 seconds? 106.8 ft

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Let $v(t) = \frac{1}{\pi} + \sin 3t$ represent the velocity of an object moving on a line. At $t = \frac{\pi}{3}$, the position is 4.

(a) Write the acceleration function. $a(t) = 3\cos 3t$

(b) Write the position function. $s(t) = \frac{t}{\pi} - \frac{1}{3}\cos 3t + \frac{2}{3}$

A particle moves along a coordinate line. Its acceleration function is $a(t)$ for $t \geq 0$. For each problem, find the position function $s(t)$ and the velocity function $v(t)$.

1) $a(t) = -2;\ s(0) = -156;\ v(0) = 25$

2) $a(t) = 6t - 40;\ s(0) = 0;\ v(0) = 100$

A particle moves along a coordinate line. Its acceleration function is $a(t)$ for $t \geq 0$. For each problem, find the position, velocity, and speed at the given value for t.

3) $a(t) = 6t - 24;\ s(0) = 0;\ v(0) = 0;$ at $t = 6$

4) $a(t) = 2;\ s(0) = 80;\ v(0) = -18;$ at $t = 6$

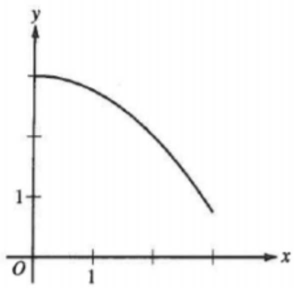
Let f be the function given by $f(x) = 4^x$. If four subintervals of equal length are used, what is the value of the left Riemann sum approximation for $\int_1^3 f(x)dx$?

(A) 30

(B) 60

(C) 62

(D) 120



Graph of f

The graph of the function f is shown above for $0 \le x \le 3$. Of the following, which has the least value?

(A) $\int_1^3 f(x)\,dx$

(B) Left Riemann sum approximation of $\int_1^3 f(x)\,dx$ with 4 subintervals of equal length

(C) Right Riemann sum approximation of $\int_1^3 f(x)\,dx$ with 4 subintervals of equal length

(D) Midpoint Riemann sum approximation of $\int_1^3 f(x)\,dx$ with 4 subintervals of equal length

(E) Trapezoidal sum approximation of $\int_1^3 f(x)\,dx$ with 4 subintervals of equal length

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If three equal subdivisions of $[-4, 2]$ are used, what is the trapezoidal approximation of

$\int_{-4}^{2} \frac{e^{-x}}{2}\,dx$?

(A) $e^2 + e^0 + e^{-2}$

(B) $e^4 + e^2 + e^0$

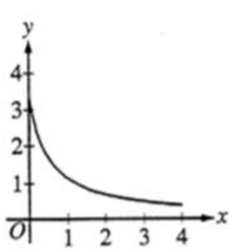
(C) $e^4 + 2e^2 + 2e^0 + e^{-2}$

(D) $\frac{1}{2}(e^4 + e^2 + e^0 + e^{-2})$

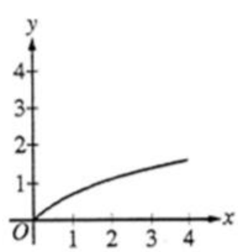
(E) $\frac{1}{2}(e^4 + 2e^2 + 2e^0 + e^{-2})$

If a trapezoidal sum overapproximates $\int_0^4 f(x)\,dx$, and a right Riemann sum underapproximates $\int_0^4 f(x)\,dx$, which of the following could be the graph of $y = f(x)$?

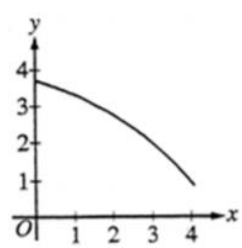
(A)



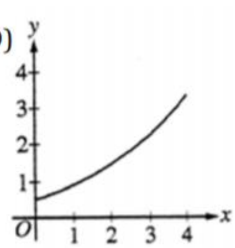
(B)



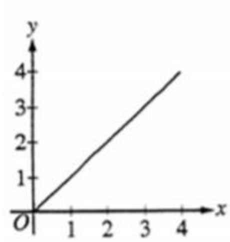
(C)



(D)



(E)



t (sec)	0	2	4	6
$a(t)$ (ft/sec²)	5	2	8	3

The data for the acceleration $a(t)$ of a car from 0 to 6 seconds are given in the table above. If the velocity at $t = 0$ is 11 feet per second, the approximate value of the velocity at $t = 6$, computed using a left-hand Riemann sum with three subintervals of equal length, is

(A) 26 ft/sec
(B) 30 ft/sec
(C) 7 ft/sec
(D) 39 ft/sec
(E) 41 ft/sec

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x	0	0.5	1	1.5	2
$f(x)$	6	14	24	28	34

The table above gives selected values for a continuous function f. If f is increasing over the closed interval $[0, 2]$, which of the following could be the value of $\int_0^2 f(x)dx$?

(A) 36
(B) 41
(C) 50
(D) 53

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t (minutes)	0	2	5	7	10
$h(t)$ (inches)	3.5	10.0	15.5	18.5	20.0

The depth of water in tank A, in inches, is modeled by a differentiable and increasing function h for $0 \le t \le 10$, where t is measured in minutes. Values of $h(t)$ for selected values of t are given in the table above.

Approximate the value of $\int_0^{10} h(t)\,dt$ using a right Riemann sum with the four subintervals indicated by the data in the table. Is this approximation greater than or less than $\int_0^{10} h(t)\,dt$? Give a reason for your answer.

$\int_0^{10} h(t)\,dt \approx 2\cdot 10 + 3\cdot 15.5 + 2\cdot 18.5 + 3\cdot 20 = 163.5$ inches

Function is increasing so right sum is an over-approximation $\Rightarrow$